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Note on Holomorphic Transformations Preserving the Bochner curvature tensor

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Abstract

In [1] D. Blair conjectured, that any holomorphic transformation of a Kaehler manifold, which preserves the Bochner curvature tensor is a homothety. He also gives a result in this direction. In this note we give the affirmative answer to the Blair's conjecture for the case $\dim_R M = 4$.

1 Preliminaries

Let M be a $2n$-dimensional Kaehler manifold with metric tensor g and complex structure J. Denote by R, S, τ the curvature tensor, the Ricci tensor (of type (1,1) as well as of type (0,2)) and the scalar curvature of M, respectively. Then the Bochner curvature tensor B is defined by

$$B(x,y)z = R(x,y)z - \frac{1}{2(n+2)}\{S(y,z)x - S(x,z)y + g(y,z)Sx - g(x,z)Sy$$

$$+S(Jy,z)Jx - S(Jx,z)Jy + g(Jy,z)SJx - g(Jx,z)SJy - 2S(Jx,y)Jz$$

$$-2g(Jx,y)SJz\} + \frac{\tau}{4(n+1)(n+2)}\{g(y,z)x - g(x,z)y + g(Jy,z)Jx$$

$$-g(Jx,z)Jy - 2g(Jx,y)Jz\} \ .$$

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It is well known, that B has the algebraic properties of the curvature tensor, see e.g. [2]. Moreover, if $\{e_i,\ Je_i\ i = 1,\dots,n\}$ is an orthonormal basis of T_pM for a point $p \in M$, then

$$\sum_{i=1}^{n} B(e_i, Je_i)x = 0 \tag{1}$$

holds good for any $x \in T_pM$, see e.g. [2].

2 Proof of Blair's conjecture for $n = 2$

Theorem 2.1. *Let M be a 4-dimensional Kaehler manifold with nonvanishing Bochner curvature tensor B. Then any holomorphic transformation f of M, which preserves B is a homothety.*

Proof: Let $x,\ y \in T_pM$, such that $B(x, Jy) \neq 0$. We choose an orthonormal basis $\{e_1, e_2, Je_1, Je_2\}$ of T_pM, which diagonalize the symmetric J-invariant endomorphism $B(x, Jy)J$, i.e.

$$B(x, Jy)Je_i = \lambda_i e_i \tag{2}$$

for $i = 1, 2$. Hence, denoting by B the Bochner curvature tensor of type (0,4) too, we have

$$B(x, Jy, Je_1, e_1) + B(x, Jy, Je_2, e_2) = \lambda_1 + \lambda_2$$

and because of (1) we obtain $\lambda_1 + \lambda_2 = 0$. For a vector x and a tensor field T on M let $\bar{x}$ and $\overline{T}$ be defined by $\bar{x} = f_* x$ and $\overline{T} = f^* T$, respectively. Since f is holomorphic and preserves the Bochner tensor, (2) implies

$$\overline{B}(\bar{x}, J\bar{y})J\bar{e}_i = \lambda_i \bar{e}_i \tag{3}$$

and hence

$$\sum_{i=1}^{2} \overline{B}(\bar{x}, J\bar{y}, J\bar{e}_i, \bar{e}_i) = \sum_{i=1}^{2} \lambda_i \bar{g}(\bar{e}_i, \bar{e}_i)\ . \tag{4}$$

On the other hand, since f preserves the Bochner tensor, (1) gives

$$\overline{B}(\bar{e}_1, J\bar{e}_1)\bar{x} + \overline{B}(\bar{e}_2, J\bar{e}_2)\bar{x} = 0$$

and consequently

$$\overline{B}(\bar{e}_1, J\bar{e}_1, J\bar{y}, \bar{x}) + \overline{B}(\bar{e}_2, J\bar{e}_2, J\bar{y}, \bar{x}) = 0\ . \tag{5}$$

From (4) and (5) we obtain $\lambda_1 \bar{g}(\bar{e}_1, \bar{e}_1) + \lambda_2 \bar{g}(\bar{e}_2, \bar{e}_2) = 0$. Since $\lambda_1 + \lambda_2 = 0$ and $(\lambda_1, \lambda_2) \neq (0, 0)$, it follows $\bar{g}(\bar{e}_1, \bar{e}_1) = \bar{g}(\bar{e}_2, \bar{e}_2)$. On the other hand, from (3) we have

$$\overline{B}(\bar{x}, J\bar{y}, J\bar{e}_i, \bar{e}_j) = \lambda_i \bar{g}(\bar{e}_i, \bar{e}_j)$$

and hence, using the algebraic properties of $\overline{B}$ and $\lambda_1 + \lambda_2 = 0$, we obtain:

$$\lambda_1 \bar{g}(\bar{e}_1, \bar{e}_2) = \overline{B}(\bar{x}, J\bar{y}, J\bar{e}_1, \bar{e}_2) = \overline{B}(\bar{x}, J\bar{y}, J\bar{e}_2, \bar{e}_1) = \lambda_2 \bar{g}(\bar{e}_1, \bar{e}_2) = -\lambda_1 \bar{g}(\bar{e}_1, \bar{e}_2)\ ,$$

i.e. $\lambda_1 \bar{g}(\bar{e}_1, \bar{e}_2) = -\lambda_1 \bar{g}(\bar{e}_1, \bar{e}_2)$. Since $\lambda_1 \neq 0$ we get $g(\bar{e}_1, \bar{e}_2) = 0$. Consequently $\bar{g} = \mu g$ at p, i.e. f is a homothety at p. Hence μ is a constant on M, because as it is easy to see, a conformal, non-homothetic change of a Kähler metric destroys the Kähler property.

References

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